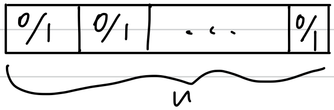
Measure Theory with Ergodic Horizons
Lecture 1

0. Motivation for measure theory.

Probability. We understand very well the probability of n coin tosses, where the probability of 1 is $p \in (0,1)$ and 0 is $1-p$.



For each $w \in 2^n := \{0,1\}^n$, the probability of n cointosses resulting in w is
$$\mathbb{P}(w) = p^{(\#1 \text{ in } w)} \cdot (1-p)^{(\#0 \text{ in } w)}$$
What if we take $n := \infty$, i.e. consider the space $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$, i.e. the set of all infinite binary sequences? Then the probability of each $w \in 2^{\mathbb{N}}$ is 0, so it's not clear how to define the probability of events in $2^{\mathbb{N}}$ that extends the finite case of 2^n.

Geometry. We would like to have a robust notion of volume for a large class of subsets of $\mathbb{R}^d$. We already know how to define volume for boxes
$$B := I_1 \times I_2 \times \dots \times I_d,$$
where $I_k \subseteq \mathbb{R}$ is an interval, namely $\text{Volume}(B) := \prod_{i=1}^{d} \ell h(I_i)$, but we would like to extend this definition so it applies to countable unions of boxes and the complements thereof. In other words, we would like the class of sets for which the volume is defined to be closed under ctbl unions/intersections and complements.

Analysis. The class of Riemann integrable functions is not closed under pointwise limits; indeed, a pointwise limit of continuous functions on $[0,1]$ is typically not even Riemann integrable. But the whole subject of analysis is based on limits, so we would like to extend the class of integrable functions so that it is closed under pointwise limits. Clearly, for a subset $B \subseteq \mathbb{R}^d$, the integral of its indicator function $\mathbb{1}_B$ will simply be the volume of B, so this task subsumes the previous goal of extending the notion of volume.

Measures, their construction, and properties

Polish spaces. A metric space (X,d) is called Polish if it is separable (there is a countable dense set) and complete (i.e. every d-Cauchy sequence converges).

Examples. (a) $\mathbb{R}$ or more generally $\mathbb{R}^n$ with metric $d_\infty(\vec{x},\vec{y}) := \|\vec{x}-\vec{y}\|_\infty := \max_{i=1}^{n} |x_i - y_i|$ for all $\vec{x},\vec{y} \in \mathbb{R}^n$. We know from analysis that d_∞ is complete, and $\mathbb{Q}^n \subseteq \mathbb{R}^n$ is dense and ctbl. Other metrics, such as $d_p(\vec{x},\vec{y}) := \|\vec{x}-\vec{y}\|_p := \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}$ for $p \geq 1$, are bi-Lipschitz equivalent to d_∞, i.e. there is a constant $C_p > 0$, depending only on p, such that $\frac{1}{C_p} d_p \leq d_\infty \leq C_p \cdot d_p$. Thus, all metric spaces $(\mathbb{R}^n, d_p)$ are Polish for $1 \leq p \leq \infty$.

(b) Closed subspaces of Polish spaces are Polish (with the same metric). What about open subsets, e.g. $(0,1)$? No with the same metric d, but maybe we can change the metric to a equivalent metric d' (i.e. d' has the same open sets as d) on $(0,1)$ that is complete. Indeed, $\underset{0 \qquad 1}{\longleftrightarrow}$ and $\underset{-\infty \qquad \infty}{\longleftrightarrow}$ look the same, in particular, they are homeomorphic, so we can copy the usual complete metric from $(-\infty,\infty)$ to $(0,1)$ through a homeomorphism. Concretely:

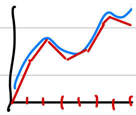
$$d'(x,y) := d(x,y) + \left| \frac{1}{d(x,\{0,1\})} - \frac{1}{d(y,\{0,1\})} \right|,$$ where for $A \subseteq \mathbb{R}$, $d(x,A) = \inf_{a \in A} d(x,a)$.

In fact, it is a theorem of Descriptive Set Theory that every G_δ set is Polishable, i.e. there is an equivalent complete metric. In particular, $[0,1)$ and $\mathbb{R} \setminus \mathbb{Q}$ are Polishable. Conversely, if a subset of a Polish space is Polishable, then it is G_δ (Alexandrov).

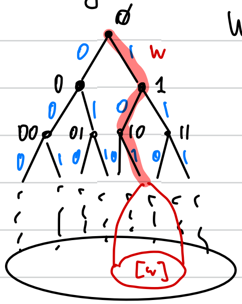
(c) The space $(C([0,1]), d_u)$ of continuous functions on $[0,1]$ with the uniform metric

$$d_u(f,g) := \max_{x\in[0,1]} |f(x)-g(x)|$$

is Polish. Indeed, it is a basic theorem in analysis that d_u is complete, and it's also separable: polynomials with rational coefficients are dense (Weierstrass), also piecewise linear functions with rational pieces and coefficients are dense.



(d) <u>The spaces $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$.</u> Let A be a ctbl nonempty set, e.g. $A = 2 = \{0,1\}$ or $A = \mathbb{N}$. Then $A^{\mathbb{N}}$ denote the space of sequences in A. We depict $A^{\mathbb{N}}$ using $A^{<\mathbb{N}}$:= the set of finite sequence in A, as a tree: picture for $A=\{0,1\}$:



We define a metric d on $A^{\mathbb{N}}$ as follows: for distinct $x,y \in A^{\mathbb{N}}$,

$$d(x,y) := 2^{-\Delta(x,y)}, \text{ where } \Delta(x,y) := \min i\in\mathbb{N}\ x_i \neq y_i.$$

For $x=y$, $d(x,y) := 0$. This indeed a metric, in fact an ultrametric (check in problem session). Also this metric is complete (HW). Furthermore, for a fixed $a_0 \in A$, the eventually a_0 sequences form a ctbl dense set. Thus, $(A^{\mathbb{N}}, d)$ is Polish.

$2^{\mathbb{N}}$ is called the Cantor space and $\mathbb{N}^{\mathbb{N}}$ is called the Baire space.

<u>The topology of $A^{\mathbb{N}}$.</u> Let $B_r(x)$ denote the open ball of radius r at x. Let $\frac{1}{2^n} < r \leq \frac{1}{2^{n-1}}$, then

$$B_r(x) := \{y\in A^{\mathbb{N}} : d(x,y) < r\}$$
$$= \{y \in A^{\mathbb{N}} : d(x,y) \leq \tfrac{1}{2^n}\} = \overline{B}_{\frac{1}{2^n}}(x)$$
$$= \{y \in A^{\mathbb{N}} : y|_n = x|_n\}.$$

For a finite word $w \in A^{<\mathbb{N}}$, the cylinder with base w is the set

$$[w] := \{y \in A^{\mathbb{N}} : y|_{|w|} = w\}$$
$$= \{wx : x \in A^{\mathbb{N}}\}.$$

We just saw that balls are cylinders and the cylinder $[w]$ is a ball centered at any element of $[w]$. Thus, cylinders are open, as well as closed: indeed the complement $[w]^c$ is a disjoint union of other cylinders:

$$[w]^c = \bigsqcup_{\substack{|w'|=|w| \\ w' \neq w}} [w'].$$

Closed and open sets are called clopen.

Proposition. $A^{\mathbb{N}}$ is compact if and only if A is finite.

Proof. HW.

In particular, the Cantor space $2^{\mathbb{N}}$ is compact, while $\mathbb{N}^{\mathbb{N}}$ isn't.

σ-algebras. Let X be a nonempty set. An algebra on X is a collection $\mathcal{A} \subseteq \mathcal{P}(X)$ of subsets of X such that $\emptyset \in \mathcal{A}$ and $\mathcal{A}$ is closed under complements and finite unions (thus also finite intersections because $A \cap B = (A^c \cup B^c)^c$).

An algebra $\mathcal{A}$ on X is called a σ-algebra if $\mathcal{A}$ is closed under countable unions (hence also cftbl intersections).

Examples. (a) Let X be a set. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be the collection of finite and cofinite (i.e. complement is finite) sets. Then $\mathcal{F}$ is an algebra (because finite union of finite sets is finite).

Let $\mathcal{S}$ be the collection ctbl and co-ctbl sets, then $\mathcal{S}$ is a σ-algebra, again because ctbl union of ctbl sets is ctbl. (HW)

(b) In a metric space X, the collection of open sets isn't typically an algebra because the complements of open sets aren't open. But the collection of clopen sets is an algebra.

In analysis, it's important to consider σ-algebras containing all open sets. The smallest one is called the σ-algebra of Borel sets.

(c) In $2^{\mathbb{N}}$, finite unions of cylinders form an algebra. (HW)

(d) In $\mathbb{R}^n$, finite unions of boxes (potentially infinite) form an algebra. (HW)